Double Least Squares Approach for Use in Structural Modal Identification

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In their procedures, several time domain modal identification algorithms encounter the solution of an overdetermined system of equations, usually by using the method of least squares. Such a method of solution is known to have statistically biased errors that can severely affect the identification accuracy, especially for the damping factors. A double least squares solution is presented and shown to reduce considerably the bias and improve the identification accuracy without a large increase in computational cost. To illustrate the proposed approach, the damping identification accuracy of the time domain modal identification algorithm referred to as the "ITD" technique is discussed. Simulated identification results show the improved accuracy of the double least squares approach when compared to the ordinary least squares method.

Introduction

MODAL identification techniques, in general, use experimentally obtained responses from a structure under test to determine its natural frequencies, damping factors, and mode shapes. The modal parameters thus obtained may be used for a wide variety of applications such as troubleshooting, responses and load prediction, control system design, dynamic modeling, and incipient failure detection.

The accuracy requirements in modal identification techniques are very much dependent on the specific application for which they are used. In some applications, the mere determination of dominant resonant frequencies, or just a qualitative analysis of the responses, may be sufficient. Others may require accurate estimation of all modal parameters of the system over a wide range of frequencies. For example, in modal identification for mathematical modeling and control system design, requirements such as high accuracy and the need for the identification technique to possess high-frequency resolution to separate closely spaced modes become crucial factors for judging the suitability of the technique for such applications. When modal analysis is used for incipient failure detection and structural integrity monitoring, the ability of the identification technique repeatedly to identify modal parameters with high accuracy in order to detect any changes in them becomes a must. In such applications, and also in flutter margins predictions, strong emphasis is needed in the determination of accurate damping factors. Another example where high accuracy is desirable is in the application of linear modal identification algorithms to nonlinear systems in a quasimodal[1] fashion to determine the types of nonlinearities.

To reduce the effect of measurements noise on the identification results, most identification algorithms implement the least squares approach to solve a set of overdetermined equations. The least squares approach is usually selected because of its simplicity and modest computational requirements. On the other hand, the least squares method is known to produce unsatisfactory results.[2] For this reason, other methods of solutions such as instrumental variable method,[3-5] maximum likelihood method,[6-8] or limited information maximum likelihood method,[9] are suggested[2] as alternative methods. Unfortunately, these methods represent expensive procedures in terms of computer cost. Naturally, the effect of a specific method used for solving an overdetermined system of equations on the identification accuracy is only partial and has to be considered in conjunction with the overall sensitivity to measurements noise of the specific identification technique.

Several recently developed time domain modal identification algorithms[10-15] use the free decay responses of a structure under test to compute a matrix whose eigenvalues and eigenvectors are related to the modal parameters of the test structure. Among the common features of these techniques is the use of the least squares approach, or the singular value decomposition, to compute the matrix of eigenvalues and eigenvectors. It has been observed that the errors in the identified damping factors are relatively large and always biased.

The correlation between the biased errors in the identified damping factors and the inherent statistically biased numerical errors in the least squares approach[16] has motivated this investigation. An approach based on performing the least squares solution twice in such a way that the biased errors are opposite to each other and thus can be reduced by averaging is suggested. The application of the proposed approach is limited in this paper to the modal identification technique referred to as the Ibrahim time domain (ITD) technique. Other techniques[14,15] can also benefit from the double least squares (DLS) approach reported in this development.

The ITD technique[10-13] has proved to be one of the promising approaches when accuracy is a high requirement. The technique is capable of accurately resolving closely spaced modes. Extremely good accuracy characterizes the technique in identifying modal parameters even when high levels of noise corrupt the responses. Only the accuracy of the identified damping factors does not match those of natural frequencies and mode shapes. Damping factors frequently show biased, but bounded errors. Reasons for such errors are explained and an algorithm is proposed to eliminate or reduce such errors.

Basic Theory of the ITD Technique

The technique uses free decay responses, containing n structural modes, to construct the $2m \times 2r$ system's response martices $[\phi]$ and $[\hat{\phi}]$ such that

$$\phi_{ij} = x_i(t_j) \tag{1}$$

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$$\phi_{ij} = x_1(t_j + \Delta t_1) \tag{2}$$

$$i = 1,2,\ldots 2m; \quad j = 1,2,\ldots 2r$$

with x_i being a physical measurement at location i on the structure under test or a physical measurement delayed Δt_2 in time that represents a pseudomeasurement.[10] The use of pseudomeasurements does not affect the identification process since modal parameters are independent of initial conditions. In constructing these response matrices, the value of m (number of degrees of freedom in the identification model) is greater than the number of modes n. This is to allow the use of an oversized identification model.[13] The number of columns in these response matrices, $2r$, is greater than $2m$ to allow for the implementation of the least squares solution.

Since the free decay responses are linear combinations of the system's modes, then the $[\phi]$ and $[\hat{\phi}]$ matrices can be written as

$$[\phi] = [\psi][\Lambda] \tag{3}$$

and

$$[\hat{\phi}] = [\psi][\alpha][\Lambda] \tag{4}$$

where $[\psi]$ is the $2m \times 2m$ matrix of complex mode shapes, $[\alpha]$ a diagonal matrix of $2m$ elements, and $[\Lambda]$ a $2m \times 2r$ matrix whose elements are

$$\Lambda_{ij} = e^{\lambda_i t_j} \tag{5}$$

The α and λ are related through the equation

$$\alpha_i = e^{\lambda_i \Delta t_1} \tag{6}$$

where λ are the characteristic roots of the system. By eliminating $[\Lambda]$ from Eqs. (3) and (4), the problem reduces to the left eigenvalue equation

$$[\alpha][\psi]^{-1}[\phi] = [\psi]^{-1}[\hat{\phi}] \tag{7}$$

To convert Eq. (7) to the more classical right eigenvalue problem

$$[A]\{\psi\} = \alpha\{\psi\} \tag{8}$$

or

$$[B]\{\psi\} = 1/\alpha\{\psi\} \tag{9}$$

the $[A]$ and $[B]$ matrices must satisfy the equations

$$[A][\phi] = [\hat{\phi}] \tag{10}$$

and

$$[B][\hat{\phi}] = [\phi] \tag{11}$$

When measurements contain no errors or noise, matrix $[B]$ is the inverse of matrix $[A]$. Since Eqs. (10) and (11) are overspecified sets of linear equations, their solutions may be obtained by using the least squares approach.

It is important to note here that the eigenvalues and eigenvectors of matrix $[A]$, for example, contain information related not only to the modal parameters of the structure under test, but also to the computational noise modes[13] representing the noise in the measurements. When noise exists and an oversized identification model is used, the $[A]$ matrix is never unique, but the eigenvalues and eigenvectors corresponding to the structural modal parameters are uniquely identified.

The purpose of the algorithm is not to identify the $[A]$ matrix, but rather its eigenvalues and eigenvectors. For that purpose, the $[B]$ matrix contain the same information. If solutions for $[A]$, or $[B]$, are to be computed from Eqs. (10) or (11) by the least squares approach, then

$$[A] = [\hat{\phi}\phi^T][\phi\phi^T]^{-1} \tag{12}$$

and

$$[B] = [\phi\hat{\phi}^T][\hat{\phi}\hat{\phi}]^{-1} \tag{13}$$

In Eqs. (12) and (13), although the matrices $[\phi]$ and $[\hat{\phi}]$ are theoretically rank deficient, $m > n$, the measurements noise completes the rank according to the theory of oversized identification models.[13] This assumption has proved to be valid for all practical cases so far.

The ITD technique, in all its previous developments and applications, was based on using the $[A]$ matrix approach [Eqs. (8), (10), and (12)]. The solution implementing the $[B]$ matrix approach was always viewed as an alternative solution to be investigated. This work is based on combining the two solutions. Prior to discussing the proposed approach, however, a discussion of the identification accuracy of the present approach is presented.

Analysis of Existing Damping Accuracy

The $[A]$ matrix, obtained from Eq. (12), yields frequencies and mode shapes with high accuracy but also shows high and biased errors in the identified damping factors. The high accuracy of the identified frequencies and mode shapes as compared to the biased errors in the identified damping factors should not confuse the fact that all these parameters are obtained from the same matrix. The difference in the accuracy is readily explained by the procedure for computing the λ from α.

If

$$\alpha_i = \beta_1 + j\gamma_i \tag{14}$$

and

$$\lambda_i = a_i + jb_i \tag{15}$$

then

$$b_i = \frac{1}{\Delta t_1}\tan^{-1}\frac{\gamma_i}{\beta_i} \tag{16}$$

$$a_i = \frac{1}{2\Delta t_1}\ln(\beta_i^2 + \gamma_i^2) \tag{17}$$

The fact is that even though the eigenvalues of the $[A]$ matrix may be of very high accuracy, the damping factors, which are related to the real part of λ, may show higher errors because of the exponential nature of Eq. (17). Actually, if ϵ is the percent error in the magnitude of the identified eigenvalue, the percent error δ in the corresponding damping factor can be shown to be

$$\delta = (100 f_a / \pi f \zeta_{ex}) \ln(1 + \epsilon/100)\% \tag{18}$$

where f_a is the aliasing frequency corresponding to the time delay Δt_1, f the frequency of the identified mode, and ζ_{ex} the exact percent damping factor. The error δ is also dependent on the phase angle of the complex eigenvalue. This dependence is implicitly expressed in Eq. (18) since the phase angle is equal to $2\pi f/f_a$.

To elaborate further on this correlation in errors, an example may be appropriate here. For a 20 Hz mode and 2% damping factor, the percent error in the identified damping factor vs the error in the magnitude of eigenvalue α is shown in Fig. 1 for aliasing frequencies of 60 and 30 Hz. From this figure and from past experiences with the ITD technique, it may be said that the errors in the computed damping factors are always bounded and, even though they may seem large, they are still a testimony to the identification accuracy of the technique as far as the frequencies and mode shapes are concerned. For example, if the error in the magnitude of the identified eigenvalue is -2%, the corresponding error in the identified damping factor will be about 96.5 and 48.2% for aliasing frequencies of 60 and 30 Hz, respectively.

This difference in accuracy may also be substantiated by the theoretical and identified parameters listed in Table 1. The two modes responses that were used to obtain these identified parameters contained 200% noise-to-signal rms ratio. By using the error correlation equation (18) and the errors in the identified damping factors, the accuracies of the identified eigenvalues' magnitudes turn to be 99.2 and 99.1% for the two modes, respectively. (This specific example[13] demonstrates the effectiveness of using an oversized identification model. A 300 degrees-of-freedom model was used to identify a 2 degrees-of-freedom system.)

Proposed Approach

The analysis in the previous section reveals that the technique under consideration has high accuracy in identifying frequencies and mode shapes. Identified damping factors show higher and biased errors due to the following reasons:

1) The solution for the matrix of eigenvalues and eigenvectors is obtained by a least squares method known to have statistically biased numerical errors.[16]

2) Even though the eigenvalues are accurately identified, the damping factors show much higher errors since they are computed from an exponential formula.

Recognizing these two facts and also considering that there exist two possible approaches to compute α and ψ, each of which uses the statistically biased least squares solution, a combination of the two solutions can be expected to produce less bias than either one of these two solutions. By denoting the $[A]$ matrix in Eqs. (8), (10), and (12) as $[A_1]$, the first solution can be summarized as

$$[A_1] = [\hat{\phi}\phi^T][\phi\phi^T]^{-1} \tag{19}$$

and

$$[A_1]\{\psi_1\} = \alpha_1\{\psi_1\} \tag{20}$$

while the second solution is

$$[B] = [\phi\hat{\phi}^T][\hat{\phi}\hat{\phi}^T]^{-1} \tag{21}$$

and

$$[B]\{\psi_2\} = \beta_2\{\psi_2\} \tag{22}$$

where

$$\beta_2 = 1/\alpha_2 \tag{23}$$

In the course of this analysis, the following assumptions are made:

1) The eigenvalues of matrix $[A_1]$ have the same order of magnitude of error as those of matrix $[B]$.

2) The errors in the identified magnitudes of the eigenvalues in both solutions have the same bias, i.e., they are either larger or smaller than the theoretical ones.

These two assumptions are justified since both $[A_1]$ and $[B]$ are computed from matrices $[\phi]$ and $[\hat{\phi}]$, which are constructed from the same experimental data. Moreover, all but two columns of $[\phi]$ and $[\hat{\phi}]$ are exactly identical. The second assumption is based on the fact that both $[A_1]$ and $[B]$ are calculated from the same computational procedures with $[\phi]$ and $[\hat{\phi}]$ interchanged; that is, the least squares approach, which is known to be statistically biased as discussed previously.

Using these assumptions, if the percent error in both α_1 and β_2 is ϵ and if the nonsubscripted quantities represent the theoretical values, then

$$\beta_2 = \beta(1+\epsilon) = (1+\epsilon)/\alpha = 1/\alpha_2 \tag{24}$$

from which

$$\alpha_2 = \alpha/(1+\epsilon) \tag{25}$$

and

$$\alpha_1 = \alpha(1+\epsilon) \tag{26}$$

$$\alpha_1 + \alpha_2 = \alpha(1+\epsilon) + \frac{\alpha}{1+\epsilon} = \alpha\,\frac{2+2\epsilon+\epsilon^2}{1+\epsilon} \tag{27}$$

By denoting α_a as the average of α_1 and α_2 then:

$$\alpha_a = \alpha(1+\epsilon^2/2 - \epsilon^3/2 + \epsilon^4/2 - \epsilon^5/2 \ldots) \tag{28}$$

It can be seen from Eq. (28) that for $\epsilon < 1$, higher-order terms are neglected and the average eigenvalue approaches that of the theoretical one. Further examination of Eqs. (27) and (28) reveals that the average eigenvalue will also have a positive bias, but the error is proportional to ϵ^2 and, by turn, is smaller than the errors in either α_1 or α_2, which are proportional to ϵ.

To simplify the computation of modal parameters, if the inverse of $[B]$ is denoted as $[A_2]$, then

$$[A_2] = [\hat{\phi}\hat{\phi}^T][\phi\hat{\phi}]^{-1} \tag{29}$$

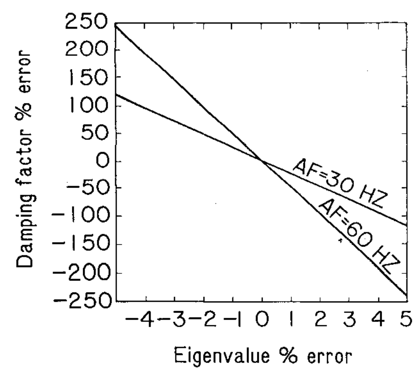

Fig. 1 Damping factor error vs eigenvalue error for a 20 Hz mode identified with 30 and 60 Hz aliasing frequency.

Table 1 Identification errors for 200% noise-to-signal ratio (single least squares approach)

Parameter	Theoretical	Identified	Error, %
f_1, Hz	20.0000	20.0062	0.03
f_2, Hz	30.0000	29.9894	0.04
ζ_1, %	1.00	2.23	123.00
ζ_2, %	1.00	1.98	98.00

Table 2 Identification errors for 200% noise-to-signal ratio (double least squares approach)

Parameter	Theoretical	Identified	Error, %
f_1, Hz	20.000	19.992	0.040
f_2, Hz	30.000	30.014	0.047
ζ_1, %	1.000	0.834	−16.600
ζ_2, %	1.000	1.135	+13.500

Table 3 Three modes, 30% noise 15 degrees-of-freedom identification results

Parameter	Theoretical	Identification SLS approach Value	Error, %	DLS approach Value	Error, %
f_1, Hz	10.0000	10.0100	0.10	10.0091	0.04
f_2, Hz	12.0000	12.0222	0.19	12.0282	0.24
f_3, Hz	15.0000	12.0092	0.06	14.9945	−0.09
ζ_1, %	2.00	3.57	19.80	1.94	−2.92
ζ_2, %	2.00	2.86	42.81	2.14	6.98
ζ_3, %	2.00	3.57	78.69	1.87	−6.58

Table 4 Three modes, 30% noise 20 degrees-of-freedom identification results

Parameter	Theoretical	Identification SLS approach Value	Error, %	DLS approach Value	Error, %
f_1, Hz	10.0000	10.0047	0.05	10.0069	0.07
f_2, Hz	12.0000	12.0355	0.30	12.0399	0.33
f_3, Hz	15.0000	15.0194	0.13	15.0116	0.08
ζ_1, %	2.00	2.32	16.00	1.93	−3.40
ζ_2, %	2.00	2.81	40.50	2.06	2.87
ζ_3, %	2.00	3.60	80.00	2.01	0.57

Table 5 Five modes, 30% noise 15 degrees-of-freedom identification results

Parameter	Theoretical	Identification SLS approach Value	Error, %	DLS approach Value	Error, %
f_1, Hz	10.0000	9.9747	−0.25	9.9976	−0.02
f_2, Hz	12.0000	12.0110	0.09	11.9893	−0.09
f_3, Hz	15.0000	15.0063	0.04	14.9965	−0.02
f_4, Hz	20.0000	20.0598	0.03	20.0494	0.25
f_5, Hz	21.0000	21.0140	0.07	21.0460	0.22
ζ_1, %	2.00	3.71	85.28	1.78	−10.99
ζ_2, %	2.00	3.85	92.59	2.06	3.11
ζ_3, %	2.00	3.14	57.13	1.69	−15.73
ζ_4, %	2.00	3.57	78.67	1.69	−15.31
ζ_5, %	2.00	3.53	76.68	1.82	−9.08

Table 6 Five modes, 30% noise 20 degrees-of-freedom identification results

Parameter	Theoretical	Identification SLS approach Value	Error, %	DLS approach Value	Error, %
f_1, Hz	10.0000	10.0170	0.17	10.0163	0.16
f_2, Hz	12.0000	11.9987	−0.01	11.9978	−0.02
f_3, Hz	15.0000	15.0082	0.05	14.9869	−0.09
f_4, Hz	20.0000	20.1145	0.57	20.0460	−0.23
f_5, Hz	21.0000	20.9165	−0.33	20.9764	−0.11
ζ_1, %	2.00	2.79	39.47	1.78	−10.81
ζ_2, %	2.00	3.24	62.10	1.90	−4.94
ζ_3, %	2.00	2.89	44.64	2.25	12.47
ζ_4, %	2.00	3.59	79.26	1.72	−13.99
ζ_5, %	2.00	3.05	52.56	1.63	−18.26

$$[A_2]\{\psi\}=\alpha_2\{\psi\} \tag{30}$$

The eigenvalues of $\frac{1}{2}[A_1+A_2]$ are $\frac{1}{2}(\alpha_1+\alpha_2)$, which are shown to be more accurate than either α_1 or α_2 and, in such a case, the matrix of eigenvalues and eigenvectors is

$$[A]=\tfrac{1}{2}([\hat{\phi}\phi^T][\phi\phi^T]^{-1}+[\hat{\phi}\hat{\phi}^T][\phi\hat{\phi}^T]^{-1}) \tag{31}$$

Using this new approach for the same case discussed in Table 1, the identified frequencies and damping factors are listed in Table 2. While the same levels of accuracy are noticed in the identified frequencies, the accuracy of the damping factors is improved. Furthermore, the usual biased errors in damping factors are noticeably reduced and the errors seem to be more random. Results in Table 2 were obtained by a smaller identification model than that used for Table 1 (65 vs 300 degrees of freedom).

Simulated Experiments

Two different sets of responses are generated and used as data for the identification program to test the proposed procedure and compare it to the previous one. The mode shapes used to generate the free decay responses are those of a simply supported beam with 10 measurements located at spacings of 1/11th of the beam's length. (For this purpose, any set of orthogonal vectors can be used as mode shapes.) The first set of data contains responses from the first three modes, while the second set has responses from five modes. Frequencies are arbitrarily assigned as 10, 12, 15, 20, and 21 Hz, while damping factors are assumed to be 2% for all modes. A sampling frequency of 100.00 Hz is used and the record length is 300 samples. Randomly generated numbers are used to simulate noise in the data and the noise-to-signal rms ratio is adjusted to 30%.

The simulated experimental responses are used to identify the system's modal parameters. Identification models of 15 and 20 modes are used for both the ordinary, or single, least squares approach [Eq. (12)] and the new proposed approach. The two approaches are here referred to as single least squares (SLS) and double least squares (DLS) solutions.

Tables 3 and 4 list the identified frequencies and damping factors for the three-mode responses and Tables 5 and 6 for the five-mode responses. Damping results obtained by using the proposed approach show noticeable improvement. Errors in damping factors seem to be more random and always less than the noise-to-signal ratio of the input data. Frequencies maintained their high identification accuracy.

Conclusions

In time domain modal identification algorithms in which least squares solutions are implemented, identified damping factors show biased errors. This is due to the inherent statistically biased numerical errors of the least squares method. When modified approach for least squares solution is utilized, the averaging of two oppositely biased solutions reduces the bias and improves the damping identification accuracy.

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